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Optimal Thrust for Maximal Rocket Turn

R. Hermesh,* S. Katzir,† and E. Kreindler‡
Technion – Israel Institute of Technology, Haifa, Israel

The problem of determining the optimal trust for maximizing a rocket's angle of turn is completely solved for the case of constant angle of attack. In the case of specified final time, there is, in general, a singular arc; the optimal combination of coast (zero thrust), boost (maximal thrust), and singular (intermediate thrust) arcs for given end conditions is determined. In the case of free final time, the optimal trajectory is a coasting arc followed by a boosting arc.

I. Introduction and Summary

CONSIDER the rocket shown in Fig. 1. The angle of attack α may be large and is assumed to be fixed. The problem is to maximize the turning angle γ at the final time t_f by optimal programming of the thrust magnitude T which is bounded by

$$0 \leq T(t) \leq T_m \tag{1}$$

The equations of motions, neglecting gravitation (which may not be permissible in some cases, but is entirely practical for our purposes), are as follows:

$$m\dot{v} = (T-D)\cos\alpha - F\sin\alpha \qquad v(0)=v_0 \qquad v(t_f)=v_f \tag{2}$$

$$mv\dot{\gamma} = F\cos\alpha + (T-D)\sin\alpha \qquad \gamma(0)=0 \qquad \gamma(t_f)\rightarrow \max. \tag{3}$$

$$\dot{m} = -T/c \qquad m(0)=m_0 \qquad m(t_f)=m_f \tag{4}$$

where m is the mass, v is the velocity, the constant c is the specific impulse, and, assuming height changes are insignificant, the aerodynamic forces F and D are proportional to v^2,

$$F = K_a v^2 \qquad D = K_d v^2 \tag{5}$$

where K_a and K_d depend on the constant nonzero angle of attack.

The above problem is one of a class of flight path optimization problems, the closest being the series of papers on minimum fuel lateral turns.[1-3] This class is characterized by the fact that the control variables, typically the thrust magnitude, appear linearly in the system equations that are nonlinear in the state variables. Consequently, the maximum principle indicates that for constraints of the type in Eq. (1), the optimal thrust either switches between $T=0$ (coast) and $T=T_m$ (boost), or it may take an intermediate value, the optimality of which cannot be answered by the maximum principle. Such an intermediate control is termed singular,[4] and its presence complicates these problems. References 1-3 are concerned with a constant altitude coordinated turn, with colinear thrust and velocity vectors; the equations of motion are therefore different from ours.

For the problem at hand it is shown in Sec. II that when the final time t_f is free, there can be no singular arc and the optimal thrust sequence is coast followed by boost (coast or boost alone are possible for special end conditions). It is shown in Sec. III that the coast-boost sequence (CB) provides also the maximal time, $t_{CB}=t_{\max}$, while a boost-coast sequence (BC) provides the minimal time, $t_{BC}=t_{\min}$. If the optimal free time $t_f^*=t_{\max.}$ is too large, one may want to specify

$$t_{\min.} \leq t_f < t_{\max.} \tag{6}$$

For a t_f specified by Eq. (6), and a final state (v_f, m_f) in the interior of the area shown in Fig. 2, it is shown in Sec. II that a singular arc must be present. In Sec. IV the generalized Legendre-Clebsch condition is shown to be satisfied along the singular arc, and the singular thrust is computed. The possible sequences of coast (C), boost (B), and singular (S) arcs are determined in Sec. V to be

$$\text{CSC} \qquad \text{CSB} \qquad \text{BSC} \qquad \text{BSB} \tag{7}$$

The dependence of the particular sequence on the end conditions (v_0, m_0) (v_f, m_f), and t_f is investigated in Sec. VI. The result is summarized in Fig. 5 and in proposition no. 7, which may be regarded as the main result of this paper. The existence of optimal control, proved in Sec. III, together with the uniqueness proved in Sec. VI, shows that the sequences of coast, boost, and singular arcs in Fig. 5 are indeed optimal.

Given the end conditions (v_0, m_0), (v_f, m_f), and t_f one can determine the optimal sequence and compute the optimal solution by the formulas in the Appendix. We have, therefore, a complete analytical solution of a nonlinear singular optimal control problem.

II. Application of the Maximum Principle

The system's Eqs. (2-5) can be written in the form

$$\dot{v} = (1/m)(T\cos\alpha - K_1 v^2) \qquad v(0)=v_0 \qquad v(t_f)=v_f \tag{8}$$

$$\dot{m} = -(1/c)T \qquad m(0)=m_0 \qquad m(t_f)=m_f \tag{9}$$

$$\dot{\gamma} = \frac{1}{m}\left(K_2 v + \frac{T\sin\alpha}{v}\right) \qquad \gamma(0)=0 \qquad \gamma(t_f)\rightarrow \max. \tag{10}$$

where

$$K_1 = K_a \sin\alpha + K_d \cos\alpha \tag{11}$$

$$K_2 = K_a \cos\alpha - K_d \sin\alpha \tag{12}$$

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*Graduate Student, Dept. of Electrical Engineering.
†Graduate Student, Dept. of Aeronautical Engineering.
‡Associate Professor, Dept. of Electrical Engineering.

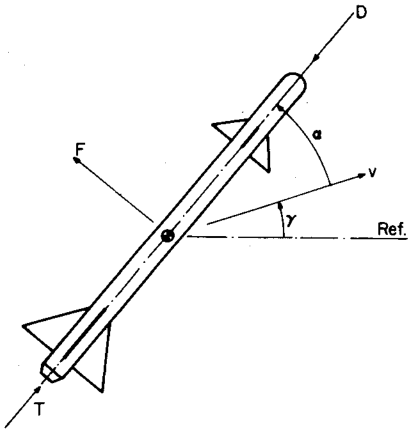

Fig. 1 Rocket problem formulation.

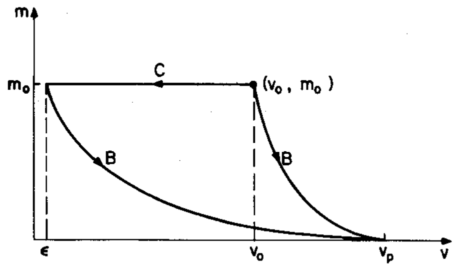

Fig. 2 Area of permissible end points (v_f, m_f).

The Hamiltonian is§

$$H=\frac{1}{m}\left(K_2 v+\frac{T\sin\alpha}{v}\right)$$
$$+p_v\frac{1}{m}(T\cos\alpha-K_1 v^2)+p_m\left(-\frac{1}{c}T\right) \tag{13}$$

where the costate variables p_v and p_m satisfy (here $H_v=\partial H/\partial v$, etc.)

$$\dot{p}_v=-H_v=\frac{T\sin\alpha}{mv^2}+p_v\frac{2K_1 v}{m}-\frac{K_2}{m} \tag{14}$$

$$\dot{p}_m=-H_m=\frac{1}{m^2}\left[\frac{T\sin\alpha}{v}+K_2 v+p_v(T\cos\alpha-K_1 v^2)\right] \tag{15}$$

Since H is linear in T, the maximization of H with respect to T depends on the coefficient which multiplies T, given by $H_T=\partial H/\partial T$,

$$H_T=\frac{\sin\alpha}{mv}+\frac{p_v\cos\alpha}{m}-\frac{p_m}{c} \tag{16}$$

Thus, T^* maximizing H is given by

$$T^*(t)=\begin{cases} 0 \quad \text{(coast)} & \text{when} \quad H_T(t)<0 \\ T_m \quad \text{(boost)} & \text{when} \quad H_T(t)>0 \\ 0\le T(t)\le T_m & \text{when} \quad H_T(t)=0 \end{cases} \tag{17}$$

If $H_T(t)$ just crosses zero, then $T^*(t)$ switches according to Eq. (17); hence, H_T is called the "switching function." (For simplicity of notation, $H(t)$ denotes $H[v(t), m(t), p_v(t), p_m(t)]$; likewise for $H_T(t)$.) If, however, on some interval

$$H_T(t)\equiv 0 \tag{18}$$

then the optimal $T(t)$ might assume intermediate values. This is the singular case.

Since H does not depend explicitly on t, an additional necessary condition of the maximum principle requires

$$H^*(t)\equiv N=\text{const} \tag{19}$$

where the star denotes that $H(t)$ is evaluated along the optimal solution. For the case where t_f is free $N=0$, i.e.,

$$H^*(t)\equiv 0 \tag{20}$$

To study the behavior of the switching function H_T, we compute $\dot{H}_T$ and obtain

$$\dot{H}_T=\frac{1}{m^2}\left[K_1\sin\alpha-K_2\left(\cos\alpha+\frac{v}{c}\right)+p_v K_1 v\left(\frac{v}{c}+2\cos\alpha\right)\right] \tag{21}$$

We write Eqs. (13), (16), and (19) as

$$H^*=N=H_T T+(v/m)(K_2-p_v K_1 v) \tag{22}$$

whence

$$p_v=\frac{(H_T T-N)m+vK_2}{v^2K_1} \tag{23}$$

which upon substitution into Eq. (21) yields

$$\dot{H}_T=\frac{1}{m^2}\left[K_a+\left(\frac{2\cos\alpha}{v}+\frac{1}{c}\right)m(H_T T-N)\right] \tag{24}$$

For the case of free t_f, $N=0$, and we can conclude at once from Eq. (24), by use of Eq. (17) and by assuming a physical problem with positive m and v, that

$$\dot{H}_T>0 \tag{25}$$

Hence:

> For the case of free t_f there can be no singular arc. Except for special end conditions ($m_f=m_0$ which results in coast only, and conditions such that $H_T\ge 0$, which results in boost only), the optimal trajectory consists of a coast arc followed by a boost arc.

When t_f is specified, the optimal trajectory must in general consist of all three arcs – coast, boost, and singular. While two given points (v_0, m_0) and (v_f, m_f) in the (v, m) plane can be joined by a trajectory consisting of two types of arcs only, the prescribed terminal time t_f will not be met unless the end conditions are specially chosen.

III. Existence of Optimal Control

For a solution to the control problem, the final specified time t_f must satisfy

$$t_{\min.}\le t_f\le t_{\max.} \tag{26}$$

assuming $t_{\min.}$ and $t_{\max.}$ exist.

For a time-optimal problem, the Hamiltonian is

$$H=p_0+\left(\frac{1}{m}p_v\cos\alpha-\frac{p_m}{c}\right)T-\frac{1}{m}p_v K_1 v^2 \tag{27}$$

§The costate p_γ is $p_\gamma=$const. For a maximum problem $p_\gamma\ge 0$. Assuming a normal problem, $p_\gamma\ne 0$, we may take $p_\gamma=1$.

Using the condition $H^* \equiv 0$ along a possible coast or singular arc, where $H_T T \equiv 0$, to obtain p_v, we arrive at

$$\dot{H}_T = p_0 \left(\frac{2 \cos\alpha}{mv} + \frac{1}{cm} \right) \tag{28}$$

Since the coefficient multiplying p_0 is positive, we conclude (assuming a normal problem $p_0 \neq 0$) that:

1) There is no singular arc in a time-optimal trajectory.

2) For minimum time, $p_0 < 0$, $\dot{H}_T < 0$, hence the optimal sequence is BC.

3) For maximum time, $p_0 > 0$, $\dot{H}_T > 0$, and the optimal sequence is CB, as in maximum turn with free t_f.

Explicit expressions for $t_{BC} = t_{min.}$ and $t_{CB} = t_{max.}$ are in the Appendix.

For given (v_0, m_0) one can delineate an area on the (v, m) plane bordered by coast and boost arcs, as shown in Fig. 2, and given by [see Eq. (A7) in the Appendix]

$$0 < m \leq m_0 \tag{29}$$

$$v_p \frac{\dfrac{v_p+\epsilon}{v_p-\epsilon}\left(\dfrac{m_0}{m}\right)^a - 1}{\dfrac{v_p+\epsilon}{v_p-\epsilon}\left(\dfrac{m_0}{m}\right)^a + 1} \leq v \leq v_p \frac{\dfrac{v_p+v_0}{v_p-v_0}\left(\dfrac{m_0}{m}\right)^a - 1}{\dfrac{v_p+v_0}{v_p-v_0}\left(\dfrac{m_0}{m}\right)^a + 1} \tag{30}$$

where v_p is given by Eq. (39) and a by Eq. (A9). Every point (v_f, m_f) in the indicated area can be attained from (v_0, m_0) by at least a CB or BC sequence on a finite time interval. Furthermore, the responses of $v(t)$, $m(t)$, and $\gamma(t)$ for any thrust $0 \leq T(t) \leq T_m$ on a finite interval $[0, t_f]$ are uniformly bounded. It then follows from a basic existence theorem[5] that *a time-optimal and a γ-optimal thrust exists for all* t_f *and* (v_f, m_f) *that satisfy Eqs. (26), (29), and (30), respectively.*

Remark: $v_f \geq \epsilon > 0$ is needed because $v_f = 0$ can be attained only by coasting with $t_f \to \infty$. Of course, low velocity makes no physical sense. For points (v_f, m_f) in the corner of Fig. 2, namely those satisfying Eq. (29) and

$$\epsilon \leq v_f \leq v_p \frac{\dfrac{v_p+\epsilon}{v_p-\epsilon}\left(\dfrac{m_0}{m_f}\right)^a - 1}{\dfrac{v_p+\epsilon}{v_p-\epsilon}\left(\dfrac{m_0}{m_f}\right)^a + 1} \tag{31}$$

there is no time-maximal or γ-maximal thrust when t_f is free. However, these points can be attained by a thrust $0 \leq T(t) \leq T_m$, and they can be included in the existence result if needed by constraining t_f with a sufficiently large upper bound.

IV. The Singular Stage

The singular case is characterized by $H_T \equiv 0$. Using this in Eq. (24) we have

$$\dot{H}_T = \frac{1}{m^2}\left[K_a - \frac{Nm}{v}\left(\frac{v}{c} + 2\cos\alpha\right)\right] \tag{32}$$

$H_T \equiv 0$ implies $\dot{H}_T \equiv 0$ so that Eq. (32) gives

$$N = \frac{K_a}{m(1/c + 2\cos\alpha/v)} \tag{33}$$

For a physical problem, v and m are positive and thus

$$N > 0 \tag{34}$$

Differentiating Eq. (32) and using $\dot{H}_T = 0$ yields

$$\ddot{H}_T = \frac{N}{m^2}\left[T\left(\frac{1}{c^2} + \frac{2\cos\alpha}{vc} + \frac{2\cos^2\alpha}{v^2}\right) - 2K_l \cos\alpha\right] \tag{35}$$

whence

$$(\ddot{H}_T)_T = \left(\frac{1}{c^2} + \frac{2\cos\alpha}{vc} + \frac{2\cos^2\alpha}{v^2}\right)\frac{N}{m^2} > 0 \tag{36}$$

In view of Eq. (36), it is evident that *the generalized Legendre-Clebsch condition*

$$(-1)^q (H_T^{(2q)})_T \leq 0 \tag{37}$$

is satisfied here in strong form with q = 1.

In the singular case $\ddot{H}_T \equiv 0$, and thus Eq. (35) gives the thrust

$$T = \frac{2K_l \cos\alpha}{(1/c + \cos\alpha/v)^2 + \cos^2\alpha/v^2} \tag{38}$$

as a function of v. For $v > 0$, we have $T > 0$. We now show that T given by Eq. (38) satisfies also $T \leq T_m$. Denote by v_p the cruise velocity under boost, obtained by setting the left side of Eq. (8) to zero

$$v_p = \sqrt{T_m \cos\alpha / K_l} \tag{39}$$

We assume that the missile always accelerates under boost. For $T = T_m$, and using Eq. (39), Eq. (8) can be written as

$$m\dot{v} = K_l (v_p^2 - v^2) \tag{40}$$

Clearly, $v \leq v_p$ with $v = v_p$ only when $m = 0$ [see also Eq. (A7)]. Then Eq. (38) gives

$$\frac{T}{T_m} = \frac{2}{(v_p/c \cos\alpha + v_p/v)^2 + (v_p/v)^2} < 1 \tag{41}$$

The relation (33) defines a one-parameter family of curves, the *singular curves,* given by

$$m = \frac{K_a}{N(1/c + 2\cos\alpha/v)} \tag{42}$$

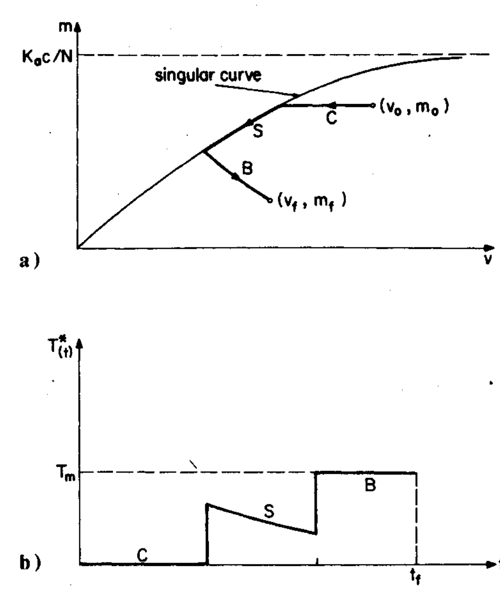

Fig. 3 a) The singular curve and a CSB trajectory. b) Profile of the optimal thrust.

along which all singular arcs must lie.

Differentiating Eq. (42) with respect to time, we obtain

$$\frac{\dot m}{\dot v}=\frac{m}{v}\,\frac{2\cos\alpha/v}{(2\cos\alpha/v+1/c)} \tag{43}$$

Since the right side of Eq. (43) is positive and $\dot m<0$, we have that $\dot v<0$, and then Eq. (38) shows that T is decreasing. Thus

> The singular thrust, given by Eq. (38), is monotonically decreasing, satisfies $0<T(t)<T_m$, and is discontinuous at junctions with coast or boost stages.

This is illustrated in Fig. 3 for a CSB trajectory.

The discontinuity of the singular thrust at the junctions is consistent with the necessary conditions in Ref. 6.

As pointed out, the singular curve depends on the constant N, which depends on the end conditions. Thus, one does not know in advance the location of the singular curve relative to the end conditions. This complicates matters. The possible sequences of C, S, and B stages are sorted out in the next section.

V. Optimal Switching

Many of the conceivable sequences of C, B, and S arcs are eliminated by the propositions below.

1) *There cannot be an intermediate coast arc.* To show this, we set $T=0$ in Eq. (24) and substitute for v its solution (A3) from the Appendix. This gives

$$\dot H_T(t)=\frac{1}{m_c^2}\left[K_a-m_cN\left(\frac{2\cos\alpha}{v_c}+\frac{1}{c}\right)-2NK_I\cos\alpha(t-t_c)\right] \tag{44}$$

which shows that $\dot H_T$ is linear in t with a negative coefficient. Now, for a switch to C from B or S at time t_c, we must have $\dot H_T(t)<0$ on some interval $(t_c,\ t_c+\delta)$, $\delta>0$. In view of the linearity of $\dot H_T(t)$, $\dot H_T$ remains negative and H_T cannot be zero again for a switch to B and S.

We next show that once $H_T(t)$ becomes positive in a switch to boost, it cannot become zero again; thus

2) *There cannot be an intermediate boost arc.* At the switch time t_b to a boost arc we have $H_T(t_b)=0$ and $\dot H_T(t_b)\geq 0$, ($\dot H_T(t_b)>0$ for CB and $\dot H_T(t_b)=0$ for SB), thus from Eq. (24)

$$K_a\geq(\cos\alpha/v(t_b)+1/c)m(t_b)N \tag{45}$$

For H_T to become zero again, it is necessary, since H_T and $\dot H_T$ are continuous during boost, that $\dot H_T(t_i)=0$ at some intermediate time t_i, There Eq. (24) gives

$$K_a=[2\cos\alpha/v(t_i)+1/c]m(t_i)[N-H_T(t_i)T_m] \tag{46}$$

From Eqs. (45) and (46)

$$N-H_T(t_i)T_m\geq\frac{m(t_b)}{m(t_i)}\left[\left(\frac{2\cos\alpha}{v(t_b)}+\frac{1}{c}\right)\Big/\left(\frac{2\cos\alpha}{v(t_i)}+\frac{1}{c}\right)\right]N \tag{47}$$

Since during boost $\dot v>0$ and $\dot m<0$, the coefficient multiplying N on the right side of Eq. (47) is larger than 1. Thus Eq. (47) gives

$$N-H_T(t_i)T_m>N \tag{48}$$

or $H_T(t_i)T_m<0$. But both $H_T(t_i)$ and T_m are positive, so we have a contradiction which proves proposition 2.

It is evident from Eq. (33) that N increases along a boost arc and decreases along a coast arc. Therefore, upon switching from a singular arc the trajectory cannot return to the same singular curve except by intermediate coast and boost arcs which are by propositions 1 and 2 not optimal. Thus,

3) *There can be at most one singular arc.* It follows now from points 1-3 that the optimal trajectory cannot have more than three arcs of type C, B, and S. If there are three, then the singular arc must be in the middle. We also have that

4) *Except for specially selected end conditions, the optimal trajectory cannot have less than three arcs of type C, B, and S.* This was pointed out already at the end of Sec. II. The given end points $(v_0,\ m_0)$ and $(v_f,\ m_f)$ can be joined by two distinct arcs (or possibly by one arc) but the time to traverse this trajectory will generally not be equal to the specified t_f. Specifically, as seen in the Appendix, the trajectory in each stage is governed by two algebraic equations, for v and for m. For an assumed two-arc trajectory, with both end points $(v_0,\ m_0,\ t_0)$ and $(v_f,\ m_f,\ t_f)$ prescribed, we get four algebraic equations with only three unknowns—namely, the coordinates of the junction point $(v_j,\ m_j,\ t_j)$. Except for special end conditions, those equations will not be simultaneously satisfied.

We come to the major conclusion that

5) *The optimal trajectory is (except for special end conditions) a sequence of exactly three arcs of the type C, B, and S, with the singular arc S in the middle. The possible three-arc trajectories are shown in Fig. 4.*

The dependence of a particular sequence on the specified end points is discussed in the next section.

VI. The Optimal Sequence

We now show that the specified end conditions $(v_0,\ m_0)$, $(v_f,\ m_f)$, and t_f uniquely determine the right sequence out of the possible ones shown in Fig. 4. Together with the existence shown in Sec. III, the uniqueness proves the optimality of the solution derived from necessary conditions.

The $(v,\ m)$ plane is covered by a one-parameter family of singular curves given by Eq. (33). Let

$$N_0=\frac{K_a}{m_0(1/c+2\cos\alpha/v_0)}\qquad N_f=\frac{K_a}{m_f(1/c+2\cos\alpha/v_f)} \tag{49}$$

The dependence of the optimal sequence on N_0, N_f, and t_f is summarized in Fig. 5, and is proven below. Explicit expressions for t_{BC}, t_{CB}, t_{SC}, t_{CS}, and t_S are given in the Ap-

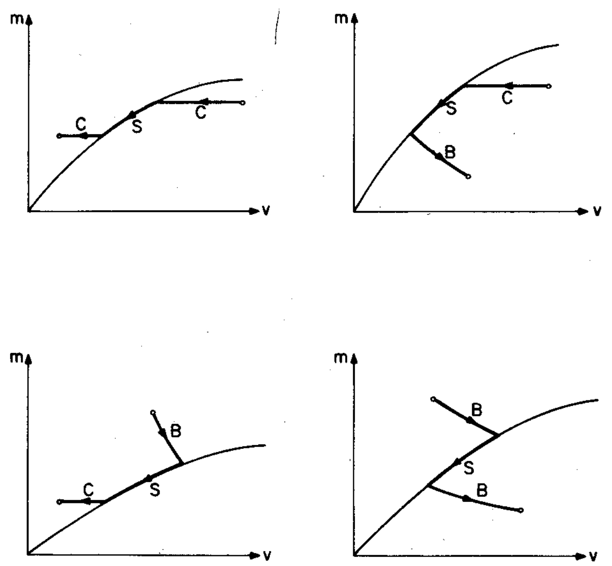

Fig. 4 All possible optimal three-arc trajectories.

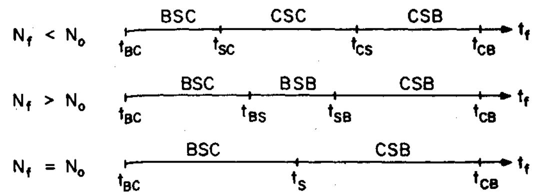

Fig. 5 Dependence of optimal sequence on N_0, N_f, and t_f.

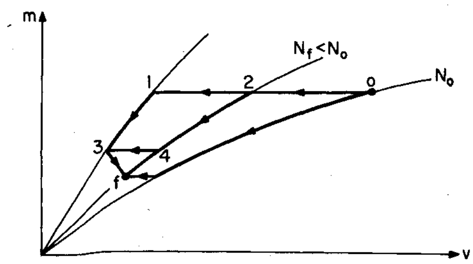

Fig. 6 Proof of proposition no. 7: the case $N_f < N_0$.

pendix. It is not possible to obtain such formulas for t_{BS} and t_{SB} (because one cannot analytically eliminate the intermediate value of m), but these times can be readily computed in a numerical example.

We note again, from Eq. (33), that N decreases along a coast arc and increases along a boost arc. Thus,¶ to continue the propositions of the previous section,

6) *If* $N_f < N_0$ *there cannot be a BSB trajectory; if* $N_f > N_0$ *there cannot be a CSC trajectory; and if* $N_f = N_0$ *there can be neither a BSB nor a CSC trajectory.* To proceed, one must consider t_f. The result is depicted in Fig. 5 and stated as follows:

7) *As the trajectory joining* (v_0, m_0) *with* (v_f, m_f) *is continuously varied from a CB trajectory to a BC trajectory, using C, B, and S arcs, the sequences are those shown in Fig. 5 and* t_f *decreases monotonically from the maximal time* t_{CB} *to the minimal time* t_{BC}. *Thus, for given end conditions* (v_0, m_0), (v_f, m_f), , *and* t_f *the optimal sequence is unique.*

This result is intuitively convincing when one graphically varies the trajectory on a (v, m)-plane covered with singular curves; it can be readily proved using the following four facts.

1) Between any two points on the (v, m)-plane, a CB trajectory is time-maximal.

2) On the other hand, a BC trajectory is time-minimal.

3) The time t_C along a coast arc between two singular curves is independent of m, i.e., t_C is equal for any such coast arcs.

4) The time t_S along a singular arc, for given m_1 and $m_2 < m_1$, varies inversely with N. Thus, for two singular arcs between the same m_1 and m_2, the time t_S for the one on the left (with a lower N) is larger.

Facts 1 and 2 are proven in Sec. III. Fact 3 follows from the formula

$$t_C = \frac{K_a}{2K_1\cos\alpha}\left(\frac{1}{N_2} - \frac{1}{N_1}\right) \qquad (N_1 > N_2) \tag{50}$$

obtained by using Eqs. (A3) and (33). Fact 4 follows from

$$t_S = \frac{m_1 - m_2}{4cK_1\cos\alpha}\left(1 + \frac{c^2K_a^2}{N^2 m_1 m_2}\right) \qquad (m_1 > m_2) \tag{51}$$

obtained by using Eq. (A13).

Proof of Proposition 7: Consider the case $N_f < N_0$ shown in Fig. 6. By fact 4, $t(1,3) > t(2,4)$, and by fact 1, $t(4,3) + t(3,f) > t(4,f)$. Adding these inequalities gives $t(1,3,f) + t(4,3) > t(2,f)$. But by fact 3, $t(4,3) = t(2,1)$ so that $t(2,1,3,f) > t(2,f)$. Adding $t(0,2)$, we have

$$t_{CSB} > t_{CS} \tag{52}$$

Using facts 3 and 4, we have at once

$$t_{CS} > t_{CSC} > t_{SC} \tag{53}$$

The remainder of the case $N_0 > N_f$,

$$t_{SC} > t_{BSC} \tag{54}$$

is proved similarly to Eq. (52).

The case $N_0 = N_f$ is quite similar to the above, and so is the proof of

$$t_{CSB} > t_{SB} \tag{55}$$

and of

$$t_{BS} > t_{BSC} \tag{56}$$

for the case $N_f > N_0$, shown in Fig. 7.

For the case $N_f > N_0$, it remains to show that

$$t_{SB} > t_{BSB} > t_{BS} \tag{57}$$

Consider Fig. 7. By fact 2, $t(0,2) > t(0,1) + t(1,2)$. By fact 7, $t(3,4) + t(4,f) > t(3,f)$. By fact 4, $t(2,4) > t(1,3)$. Adding these inequalities and using, by fact 3, $t(1,2) = t(3,4)$, yields

$$t_{SB} > t_{BS} \tag{58}$$

The central part of Eq. (57) follows at once from Eq. (58). The case in Fig. 7 when $m_4 > m_2$ is a special case. For the case $m_4 > m_2$, shown in Fig. 8, Eq. (58) is proven by drawing the intermediate singular curve through point 3. Then,

$$t(0,4) + t(4,3) > t(0,1) + t(1,3)$$

$$t(1,3) + t(3,f) > t(1,2) + t(2,f)$$

Addition gives

$$t(0,4) + t(4,f) > t(0,2) + t(2,f)$$

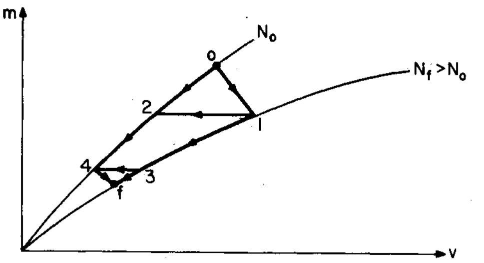

Fig. 7 Proof of proposition no. 7: the case $N_f > N_0$, $m_4 < m_2$.

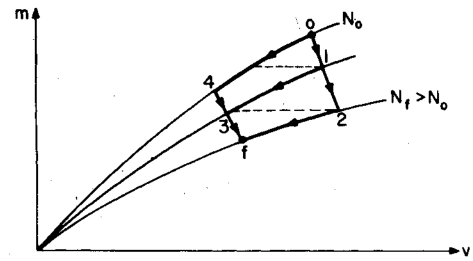

Fig. 8 Proof of proposition no. 7: the case $N_f > N_0$, $m_4 > m_2$.

¶We are indebted to the anonymous reviewer for this observation.

which shows that Eq. (58) holds for this case. In Fig. 8, $m_4 < m_1$. For the case $m_4 > m_1$, one must draw one (or more) additional intermediate singular curve (curves).

Since for each of the sequences in Fig. 5 the final time t_f was shown to be different, the optimal sequence for given end conditions is unique. This completes the proof of proposition 7.

VII. Conclusions

A complete solution of the maximization of a rocket's turn angle in a plane, using bounded thrust magnitude as the control variable, is presented here. We assume a constant angle of attack, aerodynamic forces proportional to the square of velocity, and we neglect gravity. Although only necessary conditions are invoked, optimality is proved by the existence and uniqueness of the solution.

While no new theory is developed, it is of interest that a rather complete analytical solution was possible. Given the end conditions (v_0, m_0) and (v_f, m_f), one can proceed as follows:

1) Compute $t_{\max.} = t_{CB}$ by Eq. (A18). If the final time $t_f = t_{\max.}$ is acceptable, then the optimal sequence is CB and one needs only the switch time t_b from coast to boost; compute $v(t_b)$ by Eq. (A7) and then t_b by Eq. (A5).

2) If $t_f = t_{\max.}$ is too large, select $t_f > t_{\min.}$, where $t_{\min.} = t_{BC}$ is given by Eq. (A19)

3) Use Eq. (49) to compute N_0 and N_f, to see which of the cases of Fig. 5 is applicable.

4) Let's say $N_f < N_0$. Then, compute t_{CS} and t_{SC} by Eqs. (A20) and (A21), respectively, to see which of the sequences, BSC, CSC, or CSB, is applicable.

5) Let's say $t_{SC} < t_f < t_{CS}$. Thus, the optimal sequence is CSC.

6) The optimal trajectory CSC can be determined as follows. The velocity v_s at the beginning of the singular arc can be determined from $t_{C_0} + t_S + t_{C_f} = t_f$, where v_s appears as the only unknown and where the initial and final coast times t_{C_0} and t_{C_f} are given by Eq. (A5), and t_S by Eq. (A17). The time t_{C_0} can now be determined by Eq. (A5), t_S by Eq. (A17), and the velocity v_c at the beginning of the final coast arc by Eq. (A3). The singular thrust is given by Eq. (38) with $v(t)$ by Eq. (A15).

7) Other possible sequences, depending on t_f, are computed similarly. If $N_f > N_0$, the times t_{SB} and t_{BS} must be computed (if required) numerically, as indicated at the end of the Appendix.

8) The total maximal angle of turn $\gamma^*(t_f)$ can be computed by Eqs. (A4, A8, and A16).

Appendix: Expressions for Coast, Boost and Singular Arcs, and Final Times

The system equations for $\dot{v}$ [(Eq. (8)] and for $\dot{m}$ [(Eq. (9)] can be readily integrated. The solution for $\gamma(t)$ is obtained by integrating

$$\frac{d\gamma}{dv} = \frac{K_2 v^2 + T\sin\alpha}{v(T\cos\alpha - K_1 v^2)} \tag{A1}$$

Coast

Starting at $t = t_c$ with $v(t_c) = v_c$, $m(t_c) = m_c$, and $\gamma(t_c) = \gamma_c$, we have

$$m(t) = m_c \tag{A2}$$

$$v(t) = \frac{v_c}{1 + (v_c K_1/m_c)(t - t_c)} \tag{A3}$$

$$\gamma(t) = \gamma_c + \frac{K_2}{K_1}\ln\frac{v_c}{v(t)} \tag{A4}$$

and the time t_C to a final velocity v_f is

$$t_C = \frac{m_c}{K_1}\left(\frac{1}{v_f} - \frac{1}{v_c}\right) \tag{A5}$$

Boost

Starting at $t = t_b$ with $v(t_b) = v_b$, $m(t_b) = m_b$, and $\gamma(t_b) = \gamma_b$ we have

$$m(t) = m_b - (T_m/c)(t - t_b) \tag{A6}$$

$$v(t) = v_p \frac{\dfrac{v_p + v_b}{v_p - v_b}\left(\dfrac{m_b}{m(t)}\right)^a - 1}{\dfrac{v_p + v_b}{v_p - v_b}\left(\dfrac{m_b}{m(t)}\right)^a + 1} \tag{A7}$$

$$\gamma(t) = \gamma_b + \tan\alpha\,\ln\frac{v(t)}{v_b} + \frac{1}{2}\left(\tan\alpha + \frac{K_2}{K_1}\right)\ln\frac{v_p^2 - v_b^2}{v_p^2 - v^2(t)} \tag{A8}$$

Here, v_p is the cruise velocity given by Eq. (39) and

$$a = \frac{2K_1 v_p c}{T_m} = 2c\sqrt{\frac{K_1\cos\alpha}{T_m}} \tag{A9}$$

The time t_B to $m = m_f$ is

$$t_B = (c/T_m)(m_b - m_f) \tag{A10}$$

It is useful to have the initial velocity given the end point of the boost arc; solving v_b from Eq. (A7) gives

$$v_b = v_p \frac{\dfrac{v_p + v_f}{v_p - v_f}\left(\dfrac{m_f}{m_b}\right)^a - 1}{\dfrac{v_p + v_f}{v_p - v_f}\left(\dfrac{m_f}{m_b}\right)^a + 1} \tag{A11}$$

Singular Arc

Singular arc starts at $t = t_s$ with $v(t_s) = v_s$, $m(t_s) = m_s$, and $\gamma(t_s) = \gamma_s$. In Eq. (9) for $\dot{m}$ we substitute Eq. (38) for T and eliminate v using Eq. (33). This gives

$$\dot{m} = -\frac{4cK_1\cos\alpha}{1 + (K_a c/mN)^2} \tag{A12}$$

whence

$$(m_s - m) + \left(\frac{K_a c}{N}\right)^2\left(\frac{1}{m} - \frac{1}{m_s}\right) = (t - t_s)4cK_1\cos\alpha \tag{A13}$$

and using Eq. (33) gives an equation for $m(t)$

$$[m_s - m(t)] + \left(\frac{1}{m(t)} - \frac{1}{m_s}\right)m_s^2 c^2\left(\frac{1}{c} + \frac{2\cos\alpha}{v_s}\right)^2 = (t - t_s)4cK_1\cos\alpha \tag{A14}$$

Along a singular arc, from Eq. (33)

$$m\left(\frac{1}{c} + \frac{2\cos\alpha}{v}\right) = m_s\left(\frac{1}{c} + \frac{2\cos\alpha}{v_s}\right)$$

whence

$$v(t) = \frac{2\cos\alpha}{\dfrac{m_s}{m(t)}\left(\dfrac{1}{c} + \dfrac{2\cos\alpha}{v_s}\right) - \dfrac{1}{c}} \tag{A15}$$

Substituting Eq. (38) for T in Eq. (A1) gives

$$\frac{d\gamma}{dv}=\frac{1}{2}\left(\tan\alpha-\frac{K_2}{K_1}\right)\frac{1}{v}-\frac{K_a c}{K_1 v^2}-\frac{K_a}{2K_1\cos\alpha\,(v+2c\cos\alpha)}$$

whence by integration

$$\gamma(t)=\gamma_s+\frac{1}{2}\left(\tan\alpha-\frac{K_2}{K_1}\right)\ln\frac{v(t)}{v_s}+\frac{K_a c}{K_1}\left(\frac{1}{v(t)}-\frac{1}{v_s}\right)-\frac{K_a}{2K_1\cos\alpha}\ln\frac{v(t)+2c\cos\alpha}{v_s+2c\cos\alpha} \tag{A16}$$

From Eq. (A14), the time t_S to $m=m_f$ is

$$t_S=\frac{m_s-m_f}{4cK_1\cos\alpha}\left[1+\frac{m_s}{m_f}\left(1+2\frac{c}{v_s}\cos\alpha\right)^2\right] \tag{A17}$$

Two-Arc Trajectories

Final times for two-arc trajectories are as follows:

$$t_{CB}=\frac{c(m_0-m_f)}{T_m}+\frac{m_0}{K_1 v_p}\,\frac{\dfrac{v_p+v_f}{v_p-v_f}\left(\dfrac{m_f}{m_0}\right)^a+1}{\dfrac{v_p+v_f}{v_p-v_f}\left(\dfrac{m_f}{m_0}\right)^a-1}-\frac{m_0}{K_1 v_0} \tag{A18}$$

$$t_{BC}=\frac{c(m_0-m_f)}{T_m}+\frac{m_f}{K_1 v_f}-\frac{m_f}{K_1 v_p}\,\frac{\dfrac{v_p+v_0}{v_p-v_0}\left(\dfrac{m_0}{m_f}\right)^a+1}{\dfrac{v_p+v_0}{v_p-v_0}\left(\dfrac{m_0}{m_f}\right)^a-1} \tag{A19}$$

$$t_{CS}=\frac{K_a}{2K_1\cos\alpha}\left[\frac{1}{N_f}-\frac{1}{N_0}+\frac{m_0-m_f}{2cK_a}\left(1+\frac{c^2K_a^2}{N_f m_0 m_f}\right)\right] \tag{A20}$$

$$t_{SC}=\frac{K_a}{2K_1\cos\alpha}\left[\frac{1}{N_f}-\frac{1}{N_0}+\frac{m_0-m_f}{2cK_a}\left(1+\frac{c^2K_a^2}{N_0 m_0 m_f}\right)\right] \tag{A21}$$

where N_0 and N_f are given by Eq. (49).

One cannot obtain t_{SB} and t_{BS} in closed form. Consider t_{SB}, and let v_1 and m_1 be the end point of the singular arc. Then, $t_{SB}=t_S+t_B$ where t_S and t_B are given by Eqs. (A17) and (A10), respectively, with m_f replaced by m_1. An equation for m_1 is obtained by equating v_1, given by Eq. (A15) at the end of the singular arc, with v_1 given by Eq. (A11) at the beginning of the boost arc,

$$v_1=\frac{2\cos\alpha}{\dfrac{m_0}{m_1}\left(\dfrac{1}{c}+\dfrac{2\cos\alpha}{v_0}\right)-\dfrac{1}{c}}=v_p\,\frac{\dfrac{v_p+v_f}{v_p-v_f}\left(\dfrac{m_f}{m_1}\right)^a-1}{\dfrac{v_p+v_f}{v_p-v_f}\left(\dfrac{m_f}{m_1}\right)^a+1} \tag{A22}$$

This equation for m_1 can only be solved numerically.

References

[1] Connor, M.A., "Optimization of a Lateral Turn at Constant Height," *AIAA Journal,* Vol. 5, Feb. 1967, pp. 335-338.

[2] Bryson, A.E., Jr. and Lele, M.M., "Minimum Fuel Lateral Turns at Constant Altitude," *AIAA Journal,* Vol. 7, March 1969, pp. 559-560.

[3] Vinh, N.X., "Minimum Fuel Rocket Maneuvering in Horizontal Flight," *AIAA Journal,* Vol. 11, Feb. 1973, pp. 165-169.

[4] Bell, D.J. and Jacobson, D.H., *Singular Optimal Control Problems,* Academic Press, New York, 1975.

[5] Lee, F.B. and Markus, L., *Foundations of Optimum Control Theory,* Wiley, New York, 1966, p. 262.

[6] McDanell, J.P. and Powers, W.F., "Necessary Conditions for Joining Optimal Singular and Nonsingular Subarcs," *SIAM Journal of Control,* Vol. 9, May 1971, pp. 161-173.